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Coherence measures and their relation to fuzzy similarity and inconsistency in knowledge bases

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Abstract Intuitively it seems that the coherence of information received from heterogeneous sources should be one factor in determining the reliability or truthfulness of the information, yet the concept of coherence is extremely difficult to define. This paper draws on recent work on probabilistic measures of coherence by investigating two measures with contrasting properties and then explores how this work relates to similarity of fuzzy sets and comparison of knowledge bases in cases where inconsistency is present. In each area contrasting measures are proposed analogous to the probabilistic case. In particular, concepts of fuzzy and logical independence are proposed and in each area it is found that sensitivity to the relevant concept of independence is a distinguishing feature between the contrasting measures. In the case of inconsistent knowledge bases, it is argued that it is important to take agreeing information and not just conflicting and total information into account when comparing two knowledge bases. One of the measures proposed achieves this and is shown to have a number of properties which enable it to overcome some problems encountered by other approaches.

Keywords Agreement · Coherence · Fuzzy similarity · Inconsistency · Independence · Probability

1 Introduction

In situations where information is available from different sources it can be useful to know how coherent the information is. For example, we might be more inclined to accept the testimony of a witness if her story coheres with that of other witnesses or with information obtained in other ways. But what exactly is coherence? One point to note is that it is not the same as consistency since coherence seems to involve some degree of agreement or support

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among the pieces of information in question. While this much can be granted, no generally accepted definition of coherence has been forthcoming.

Nevertheless, in recent years considerable attention has been given to characterising coherence in probabilistic terms (see, for example, Shogenji 1999; Olsson 2002; Bovens and Hartmann 2003; Fitelson 2003). The main focus of such work has been on obtaining a suitable measure of coherence and on determining whether a greater degree of coherence implies a higher probability of the information being true. Detailed discussions on these matters as well as discussions relating coherence to such topics as reliability, confirmation, testimony and theory choice in science have been explored in some detail by Bovens and Hartmann (2003) and Olsson (2005). Here the focus is different since the goal of the paper is to investigate the application of the concept of coherence in other areas, rather than addressing questions about the fundamental nature of coherence. To achieve this, we present a brief description of probabilistic measures of coherence by considering two particular measures and some significant differences between them, before going on to investigate how coherence might be related to similarity in fuzzy sets and inconsistency in knowledge bases.

As in the case of probabilistic coherence, two contrasting accounts of fuzzy similarity are provided. The first of these gives an account of similarity that does not take the degree of fuzziness into account. While many similarity measures have been proposed in the literature (see, for example, Chen et al. 1995; Wang et al. 1995; and references therein) most fall into this category since they satisfy the condition that the similarity of any fuzzy set with itself should be one. However, not all similarity measures satisfy this condition and indeed Sancho-Royo and Verdegay (2000, 2005) have proposed what they call a *coherence* measure to incorporate both similarity (which they take to satisfy the condition noted) and fuzziness. In this paper, we provide necessary conditions for a second type of similarity measure that does take the degree of fuzziness into account. Properties of the two concepts of similarity, analogous to the two concepts of probabilistic coherence, are explored and it is shown how measures of the second type can be generated from measures of the first type. It turns out that the second type of similarity measure is closely related to Sancho-Royo and Verdegay's concept of coherence.

The next part of the paper applies the concept of coherence to knowledge bases, with the main focus being on cases where two knowledge bases are in conflict with each other. Considerable attention has been given to dealing with inconsistency (see, for example, Gärdenfors 1988; Benferhat et al. 1997; Hunter 2000; Priest 2002) and there has also been work proposing measures of information and inconsistency for knowledge bases (Lozinskii 1994; Knight 2001, 2003), which in some cases are used to order sources of information (Hunter 2002; Konieczny et al. 2003). Suppose information is received from various sources and that in each case there is some inconsistency with domain knowledge. In such cases it may be appropriate to use a measure of inconsistency to obtain an ordering of the sources, with the source that is least inconsistent with the domain knowledge considered the most reliable. However, in many cases it is not only the degree of inconsistency that is relevant, but also the amount of information provided by the source. If any source provides a sufficient amount of useful information this may be enough to compensate for its inconsistency, thus a trade-off is required. See Hunter and Konieczny (2004) for discussion on this point.

However, a further consideration suggests that in some cases information in agreement between two knowledge bases, in addition to information in conflict and the total information, may be important. Suppose we wish to order information from various sources K_i in terms of their relationship with another knowledge base K. One approach is to consider the union of information from each source with K, obtaining $K_i \cup K$, and to calculate the quantity of information and inconsistency in $K_i \cup K$. Consider a simple case where $K = \{p, q\}$, $K_1 = \{\neg p\}$

and $K_2 = \{\neg p, q\}$. The problem here is that in taking the union this approach fails to distinguish between K_1 and K_2. Suppose, for example, we are considering multiple witness reports. Although we might expect there to be some degree of conflict between any two reports, it may be that a high degree of coherence between any pair of reports will be crucial in accepting their authenticity. As pointed out by Qi et al. (2005), the notion of agreement is also relevant in terms of interactions between agents since agreement could offset conflict between them. The general aim of this paper is to propose a measure of coherence which incorporates agreement, inconsistency and information of two knowledge bases.

The work of Qi et al. is highly relevant to the current paper since they discuss two types of agreement, weak and strong, with the former corresponding to information provided by one source and not the other, while the latter corresponds to information provided by both sources. These concepts are then used to define a degree of conflict, a degree of strong agreement and a degree of weak agreement. Although their work focussed on prioritised knowledge bases, it is still relevant for the non-prioritised case. Their work will be discussed further in Sect. 4, including possible improvements that result from using a single measure of coherence.

The structure of the paper is as follows. Section 2 introduces the concept of coherence in a probabilistic context and presents a discussion of the contrasting properties of two different coherence measures. Section 3 focuses on the similarity of fuzzy sets and provides a discussion of two different conceptions of fuzzy sets that parallels the discussion of probabilistic coherence measures. Section 4, considers how the concept of coherence might be relevant to comparing knowledge bases. First, we look at two coherence measures of a pair of knowledge bases, which are individually consistent as well as being consistent with each other. Once again, this discussion parallels the probabilistic case. It also provides a foundation for dealing with the more interesting problem of two knowledge bases which are inconsistent with each other. Finally, Sect. 5 presents some conclusions.

2 Probabilistic measures of coherence

Although there is no agreed definition of coherence for a set of beliefs, there is agreement about some of the features such a measure should possess. Here the focus is on the case of two beliefs[1] since there is no agreement even in this case and since this problem will be the most relevant for relating coherence to other areas in the remaining sections of this paper. The following three points are widely, although not universally (see, for example, Shogenji 1999), accepted as necessary conditions for the coherence of two beliefs A and B, denoted $C(A, B)$,

1. $C(A, B) = C(B, A)$,
2. $C(A, B)$ is maximal if A and B are logically equivalent, and
3. $C(A, B)$ is minimal if A and B are logically inconsistent.

A simple measure, C_1, for the coherence of two beliefs satisfying these conditions has been discussed by Olsson (2002) and Glass (2002) and is defined as follows,

$$C_1(A, B) = \frac{P(A \wedge B)}{P(A \vee B)}, \tag{1}$$

[1] A problem for probabilistic measures of coherence concerns how the beliefs should be individuated. The problem is that the degree of coherence of a belief set depends on how the beliefs have been individuated so that, for example, the coherence of three beliefs A, B and C is not in general the same as the coherence of the beliefs $A \wedge B$ and C. Shogenji (2001) has argued that beliefs should be individuated by their sources, although this has been criticised by Moretti and Akiba (2007).

whenever $P(A \vee B) \neq 0$, where P is a probability distribution.

Example 1 Suppose an unbiased die is rolled and consider the belief, A, that the result will be even and the belief, B, that the result will be greater than three. The coherence of A and B according to C_1 is $1/2$.

In addition to satisfying the three criteria noted above C_1 also possesses the following properties.

Proposition 1 *For probability distributions P and P',*

(*a*) *if $P(A|B) > P'(A|B)$ and $P(B|A) > P'(B|A)$, then A and B are more coherent according to C_1 on distribution P than on distribution P' (Bovens and Olsson 2000);*
(*b*) *if $P(A|B) = P'(A|B)$ and $P(B|A) = P'(B|A)$, then A and B are equally coherent according to C_1 on distribution P and distribution P'.*

Proof Trivial if $P(A \wedge B)$. Otherwise, dividing numerator and denominator by $P'(A \wedge B) = 0$ and using Bayes' theorem we note that C_1 can be written as

$$\begin{aligned} C_1(A, B) &= \frac{P(A \wedge B)}{P(A) + P(B) - P(A \wedge B)} \\ &= \left[\frac{1}{P(A|B)} + \frac{1}{P(B|A)} - 1\right]^{-1}. \end{aligned} \tag{2}$$

(a) if $P(A|B) > P'(A|B)$ and $P(B|A) > P'(B|A)$, it follows that $1/P(A|B)+1/P(B|A)-1 < 1/P'(A|B) + 1/P'(B|A) - 1$ and so $C_1(A, B)$ is greater for distribution P than it is for P'.
(b) The proof follows directly from Eq. 2.

□

Proposition 1 captures the idea that it is the conditional probability of each belief given the other that is important. The important factor for the C_1 measure is the degree of overlap between the beliefs rather than how probable they are in the first place, i.e. if the relevant conditional probabilities are the same for the distributions P and P' the coherence according to C_1 will be the same irrespective of the marginal probabilities of A and B. Although Proposition 1 seems quite plausible, it is not generally considered to be a necessary requirement for a coherence measure. In fact, even Bovens and Olsson (2000) who propose part (a) of the proposition draw attention to some problems with it (see also Shogenji 1999). Presently we shall consider another measure of coherence proposed in the literature which does not satisfy Proposition 1. First, we consider another property of C_1.

Proposition 2 *For any belief C, $C_1(A \vee C, B \vee C) \geq C_1(A, B)$.*

Proof We note that in going from $C_1(A, B)$ to $C_1(A \vee C, B \vee C)$ the numerator is increased more than the denominator as the following expression shows,

$$C_1(A \vee C, B \vee C) = \frac{P(A \wedge B) + P(C \wedge \neg A \wedge \neg B) + P(C \wedge A \wedge \neg B) + P(C \wedge \neg A \wedge B)}{P(A \vee B) + P(C \wedge \neg A \wedge \neg B)}. \tag{3}$$

From this expression the proof follows immediately. □

Proposition 2 is analogous to a property shared by many measures of the similarity of two fuzzy sets as will be discussed further in Sect. 4.

Relating coherence to the notion of support Fitelson (2003) defines a measure of coherence in terms of the following measure of support (which is based on Kemeny and Oppenheim 1952).

Definition 1 (*Kemeny and Oppenheim 1952*; *Fitelson 2003*) A measure of support, F, which B gives to A can be defined as

$$F(A, B) = \frac{P(B|A) - P(B|\neg A)}{P(B|A) + P(B|\neg A)},$$

if A is contingent and B is not a necessary falsehood. $F(A, B)$ is defined to be 1 if A and B are necessary truths, 0 if A is a necessary truth and B is contingent, and -1 if B is a necessary falsehood.

This measure of support is then used to define a coherence measure which, in the case of two beliefs A and B, is given by

$$C_2(A, B) = \frac{1}{2}\{F(A, B) + F(B, A)\}. \tag{4}$$

The C_2 measure satisfies the three requirements noted earlier for a coherence measure and thus has much in common with C_1 even though C_2 is defined on the interval $[-1, 1]$ rather than $[0, 1]$. Nevertheless, there are also some very significant differences between C_1 and C_2. In particular, Fitelson's measure, C_2, is constructed to be sensitive to probabilistic dependence so that $C_2(A, B) > 0$ if A and B have a positive probabilistic dependence on each other (i.e. $P(A|B) > P(A|\neg B)$), $C_2(A, B) < 0$ if A and B have a negative probabilistic dependence, and $C_2(A, B) = 0$ if A and B are probabilistically independent. Thus, zero provides a *neutral* point distinguishing positive and negative dependence. A consequence of taking account of probabilistic dependence in this way is that neither Proposition 1 (see Glass 2005) nor Proposition 2 hold for the C_2 measure. The following proposition emphasizes this difference between the two measures by pointing out the insensitivity of C_1 to probabilistic dependence.

Proposition 3 *The C_1 coherence measure defined in Eq. 1 is insensitive to the probabilistic dependence in the sense that* $\forall \varepsilon \in (0, 1)$

(*a*) *there are beliefs A and B that have a negative probabilistic dependence and* $C_1(A, B) > 1 - \varepsilon$,

(*b*) *there are beliefs A and B that have a positive probabilistic dependence and* $C_1(A, B) < \varepsilon$.

Proof

(a) Let A represent the belief that a number, randomly selected from the interval $[0, 1]$ will be in the interval $[0, r]$, where $1/2 < r < 1$. Similarly, let B represent the belief that it will be in the interval $[1 - r, 1]$. We note that A and B are negatively dependent since $P(A|B) = (2r - 1)/r < P(A|\neg B) = 1$. Now, let $r = 1 - \varepsilon/4$. We find that $C_1(A, B) = 1 - \varepsilon/2 > 1 - \varepsilon$.

(b) Again we consider a number selected randomly from the interval $[0, 1]$, but we now take A to be the belief that it will be in the interval $[0, 1/r]$, where $r > 2$, and B the belief that it will be in the interval $[0, 1/2]$. We note that A and B are positively dependent since $P(A|B) = 2/r > P(A|\neg B) = 0$. Now, select a value of r such that $r > 2/\varepsilon$. We find that $C_1(A, B) = 2/r < \varepsilon$. □

Clearly, the C_2 measure does not satisfy a corresponding version (taking into account its different range) of Proposition 3. Thus, Propositions 1, 2 and 3 provide ways of distinguishing C_1 and C_2 and, more generally, ways of distinguishing very different conceptions of coherence.

3 Similarity of fuzzy sets

This section investigates the relationship between the probabilistic measures of coherence as discussed in Sect. 2 and similarity measures for fuzzy sets. In what follows we shall consider a finite universe of discourse, X, and a fuzzy set, A, defined on X, represented by the set of pairs $\{(x, \mu_A(x)), x \in X\}$. The complement of a fuzzy set A, denoted $\overline{A}$, is defined by $\{(x, 1 - \mu_A(x)), x \in X\}$. We also define I to be the fuzzy set such that $\mu_I(x) = 0.5\,\forall x \in X$. For a detailed account of fuzzy set theory we refer the reader to Dubois and Prade (1980).

3.1 Two conceptions of fuzzy similarity

Considering two fuzzy sets A and B, the following points provide necessary conditions for a measure S to be a similarity measure.

1. $S(A, B) = S(B, A)$,
2. $S(A, B)$ is maximal if A and B are identical crisp sets, and
3. $S(A, B)$ is minimal if A and B are crisp and $A = \overline{B}$.

While these conditions are reasonable as far as they go, more precise formulations can be given to distinguish particular types of similarity measures. For example, as the comparison of Chen et al. (1995) illustrates, a number of measures satisfy conditions which we shall use to define a *type one* similarity measure.

Definition 2 A similarity measure is said to be a type one similarity measure iff:

1. $S(A, B) = S(B, A)$,

2′. $S(A, B)$ is maximal iff A and B are identical, and

3′. $S(A, B)$ is minimal iff $|A \cap B| = 0$.

A commonly used type one similarity measure is given by

$$S_1(A, B) = \frac{|A \cap B|}{|A \cup B|}, \tag{5}$$

provided $|A \cup B| \neq 0$. In the case where $|A \cup B| = 0$ (i.e. $A = B = \emptyset$) we define $S_1(A, B) = 1$. Clearly, S_1 corresponds with the C_1 coherence measure and shares some of its properties. For example, the following proposition corresponds with Proposition 2.

Proposition 4 *For any fuzzy set C, $S_1(A \cup C, B \cup C) \geq S_1(A, B)$.*

Proof See the proof of Proposition 2.1 in Wang et al. (1995).

It should be noted, however, that not all type one similarity measures satisfy this proposition.

By way of contrast to type one similarity measures, we now consider a second way of making conditions 1, 2 and 3 more precise by defining a *type two* similarity measure.

Definition 3 A similarity measure is said to be a type two similarity measure iff:

1. $S(A, B) = S(B, A)$,
2″. $S(A, B)$ is maximal iff A and B are identical crisp sets, and
3″. $S(A, B)$ is minimal iff A and B are crisp and $A = \overline{B}$.

Note that 2″ and 3″ provide *necessary* and sufficient conditions for maximality and minimality respectively and so differ from conditions 2 and 3 presented earlier, which were only sufficient conditions. While type two measures are not at all common in the literature, they enable the degree of fuzziness to be taken into account and provide a link with the probabilistic measures of coherence. An example of a type two measure, which provides a contrast to the S_1 measure, has been defined by Dubois and Prade (1980) and can be expressed as follows,

$$S_2(A, B) = 1 - \frac{\sum_x max[min(\mu_A(x), 1 - \mu_B(x)), min(1 - \mu_A(x), \mu_B(x))]}{|X|}. \quad (6)$$

Unlike S_1, S_2 does not satisfy Proposition 4 as illustrated by the following example.

Example 2 Let $A = (0.4, 0.3)$, $B = (0.3, 0.1)$ and $C = (0.5, 0.4)$ be three fuzzy sets. $S_2(A \cup C, B \cup C) = 0.55 < 0.65 = S_2(A, B)$. By contrast $S_1(A \cup C, B \cup C) = 1 > 0.57 = S_1(A, B)$.

This suggests an analogy between the similarity measures S_1 and S_2 and the coherence measures C_1 and C_2 respectively, which can be made stronger by considering the notion of *weak equality* as discussed by Dubois and Prade. Two fuzzy sets A and B are said to be in weak equality if $\mu_A(x)$ and $\mu_B(x)$ are both greater than or equal to $1/2$ or both less than or equal to $1/2$ for all x. It turns out that $S_2(A, B) \geq 1/2$ if and only if A and B are in weak equality, while no such dependence exists in the case of the S_1 measure. Thus, the notion of weak equality plays a similar role for the S_2 measure as probabilistic dependence does for the C_2 measure. We also note that for any fuzzy set A, $S_2(A, I) = S_2(I, I) = 1/2$, where $I = \{(x, 1/2), x \in X\}$. Thus, S_2 takes fuzziness into account since the similarity of a set with itself depends on the fuzziness of the set.

This point also provides a key difference between type one and type two measures more generally. Type one measures do not depend on the degree of fuzziness of the sets in the sense that the similarity of a set with itself is one irrespective of its degree of fuzziness. By contrast type two measures are able to take the degree of fuzziness into account. As the following proposition shows, any type one similarity measure can be used to construct a type two similarity measure, which is analogous in some respects to the C_2 probabilistic coherence measure.

Proposition 5 *For any type one similarity measure, S, with range* $[0, 1]$, *define a new measure* S_{new} *with range* $[-1, 1]$ *as follows,*

$$S_{new}(A, B) = \frac{1}{2}\Big[\Big(S(A, B) + S(\overline{A}, \overline{B})\Big) - \Big(S(A, \overline{B}) + S(\overline{A}, B)\Big)\Big]. \quad (7)$$

S_{new} *satisfies the following properties:*

(a) S_{new} *is a type two similarity measure,*
(b) *For any fuzzy set* A, $S_{new}(A, I) = 0$,
(c) *It is not necessarily the case that for a fuzzy set* C, $S_{new}(A \cup C, B \cup C) \geq S_{new}(A, B)$.

Proof (a) S_{new} is a type two similarity measure

(i) $S_{new}(A, B) = S_{new}(B, A)$. This follows from the fact that S is symmetric and the definition of S_{new}.

(ii) $S_{new}(A, B)$ is maximal (=1) iff A and B are identical crisp sets.
Sufficiency. Suppose A and B are identical crisp sets. It follows from the fact that S satisfies condition $2'$ that $S(A, B) = S(\overline{A}, \overline{B}) = 1$. Since S also satisfies condition $3'$ it follows that $S(A, \overline{B}) = S(\overline{A}, B) = 0$, and so $S_{new}(A, B) = 1$.
Necessity. If $S_{new}(A, B) = 1$, then $S(A, \overline{B}) = S(\overline{A}, B) = 0$. From the fact that S satisfies condition $3'$, it follows that $\forall x \in X\ min(\mu_A(x), 1 - \mu_B(x)) = min(1 - \mu_A(x), \mu_B(x)) = 0$ and so $\forall x$ either $\mu_A(x) = \mu_B(x) = 0$ or $\mu_A(x) = \mu_B(x) = 1$ and so A and B are identical crisp sets.

(iii) $S_{new}(A, B)$ is minimal (=−1) iff A and B are crisp and $A = \overline{B}$.
Sufficiency. Suppose A and B are crisp sets and $A = \overline{B}$. It follows from the fact that S satisfies condition $3'$ that $S(A, B) = S(\overline{A}, \overline{B}) = 0$. Since S also satisfies condition $2'$ it follows that $S(A, \overline{B}) = S(\overline{A}, B) = 1$, and so $S_{new}(A, B) = -1$.
Necessity. If $S_{new}(A, B) = -1$, then $S(A, B) = S(\overline{A}, \overline{B}) = 0$ and $S(A, \overline{B}) = S(\overline{A}, B) = 1$. Since S satisfies condition $2'$ it follows that $A = \overline{B}$ and so $\forall x \in X\ \mu_A(x) = 1 - \mu_B(x)$. Furthermore, since S satisfies condition $3'$, it follows that $|A \cap B| = 0$ and so $\forall x \in X\ min(\mu_A(x), \mu_B(x)) = 0$. Hence, $\forall x$ either $\mu_A(x) = 0$ and $\mu_B(x) = 1$ or $\mu_B(x) = 0$ and $\mu_A(x) = 1$ and so A and B are crisp sets.

(b) Since $I = \{(x, 1/2), x \in X\}$, $I = \overline{I}$. Thus, $S(A, I) = S(A, \overline{I})$ and $S(\overline{A}, I) = S(\overline{A}, \overline{I})$, and so $S_{new}(A, I) = 0$.

(c) Suppose S_{new} is obtained from the S_1 measure via Eq. 7 and let A, B and C be the sets defined in Example 1. We obtain $S_{new}(A, B) = 0.32 > 0.18 = S_{new}(A \cup C, B \cup C)$. □

The analogous nature of the relationship between a measure S_{new} as defined in Eq. 7 and the C_2 measure can be seen by comparing Proposition 5(c) with the fact that C_2 does not satisfy Proposition 2. Furthermore, S_{new} has been constructed in such a way that the neutral point where $S_{new} = 0$ becomes more significant as it is in the case of C_2. To bring this out more clearly, the terms in Eq. 7 can be used to define the fuzzy-(in)dependence of two fuzzy sets as follows,

Definition 4 For any similarity measure, S, and for two fuzzy sets A and B we say that

(i) A and B are positively fuzzy-dependent iff $S(A, B) + S(\overline{A}, \overline{B}) > S(A, \overline{B}) + S(\overline{A}, B)$,
(ii) A and B are fuzzy-independent iff $S(A, B) + S(\overline{A}, \overline{B}) = S(A, \overline{B}) + S(\overline{A}, B)$,
(iii) A and B are negatively fuzzy-dependent iff $S(A, B) + S(\overline{A}, \overline{B}) < S(A, \overline{B}) + S(\overline{A}, B)$.

It should be noted that this definition of (in)dependence is relative to the similarity measure, S, under consideration. By using this definition, we now have the following proposition which corresponds to Proposition 3.

Proposition 6 *Any type one similarity measure, S, is insensitive to fuzzy-dependence between sets as defined in Definition 4 in the sense that*

(a) *there are sets A and B which are fuzzy-independent and* $S(A, B) = 0$ (*i.e. has its minimum value*),
(b) *there are sets A and B which are fuzzy-independent and* $S(A, B) = 1$.

Proof

(a) Consider the fuzzy sets $A = \emptyset$ and $B = I$. By condition $3'$ and the fact that $B = \overline{B}$, we know that $S(A, B) = S(A, \overline{B}) = 0$ and that $S(\overline{A}, \overline{B}) = S(\overline{A}, B)$ and so A and B are fuzzy-independent for measure S.
(b) Consider the fuzzy sets $A = B = I$. By condition $2'$ we know that $S(A, B) = 1$, yet A and B are fuzzy-independent for measure S. □

For the S_1 similarity measure defined in Eq. 5, the following proposition more directly parallels Proposition 3,

Proposition 7 *The S_1 similarity measure defined in Eq. 5 is insensitive to fuzzy-dependence between sets as defined in Definition 4 in the sense that* $\forall \varepsilon \in (0, 1)$

(*a*) *there are sets A and B which have a negative fuzzy-dependence and* $S_1(A, B) > 1 - \varepsilon$,
(*b*) *there are sets A and B which have a positive fuzzy-dependence and* $S_1(A, B) < \varepsilon$.

Proof

(a) Consider the fuzzy sets $A = (\frac{1}{2} - \frac{\varepsilon}{4})$ and $B = (\frac{1}{2} + \frac{\varepsilon}{4})$. Since $S_1(A, B) = S_1(\overline{A}, \overline{B}) < 1$ and $S_1(\overline{A}, B) = S_1(A, \overline{B}) = 1$, A and B are negatively fuzzy-dependent. Furthermore, $S_1(A, B) = (\frac{1}{2} - \frac{\varepsilon}{4})/(\frac{1}{2} + \frac{\varepsilon}{4}) = 1 - \frac{2\varepsilon}{2+\varepsilon} > 1 - \varepsilon$.
(b) Consider the fuzzy sets $A = \{\frac{\varepsilon}{4}\}$ and $B = \{\frac{1}{2} - \frac{\varepsilon}{4}\}$. Since $S_1(A, B) = \frac{\varepsilon}{2-\varepsilon} > \frac{\varepsilon}{2+\varepsilon} = S_1(A, \overline{B})$ and $S_1(\overline{A}, \overline{B}) = \frac{2+\varepsilon}{4-\varepsilon} > \frac{2-\varepsilon}{4-\varepsilon} = S_1(\overline{A}, B)$, A and B are positively fuzzy-dependent. Furthermore, $S_1(A, B) = \frac{\varepsilon}{2-\varepsilon} < \varepsilon$. □

3.2 Discussion of related work

It is clear from the foregoing discussion that a key difference between type one similarity measures and type two similarity measures is that the latter can take the degree of fuzziness into account while the former cannot. For example, for measures constructed using Eq. 7 similarity is maximal for identical crisp sets, but non-maximal for identical non-crisp sets, e.g. $S_{new}(I, I) = 0$, where $I = \{(x, 1/2), x \in X\}$. It is interesting to note that Sancho-Royo and Verdegay (2000, 2005) propose a type of measure to take into account both similarity and degree of fuzziness, which they call a *coherence measure*. Their definition of a coherence measure can be stated as follows,

Definition 5 Let A and B be finite fuzzy sets. A function $Cohe$ such that $Cohe(A, B)$ lies in the interval [0, 1] is a coherence measure iff:

(i) $Cohe(A, B) = Cohe(B, A)$,
(ii) $Cohe(A, \overline{B}) = 1 - Cohe(A, B)$,
(iii) $Cohe(\emptyset, X) = 0$, where X is the crisp set such that $\mu_X(x) = 1\ \forall x \in X$.

The following proposition establishes the relationship between this definition of a coherence measure and the foregoing discussion of similarity measures.

Proposition 8 *For any type one similarity measure, S, consider the type two similarity measure S_{new} defined in Eq. 7. This similarity measure can be used to generate a coherence measure as follows,*

$$Cohe(A, B) = \frac{1}{2}(S_{new}(A, B) + 1). \tag{8}$$

Proof

(i) $Cohe(A, B) = Cohe(B, A)$.
Follows from the fact that S is symmetric.

(ii) $Cohe(A, \overline{B}) = 1 - Cohe(A, B)$.
$Cohe(A, \overline{B}) = \frac{1}{4}\Big[\Big(S(A, \overline{B}) + S(\overline{A}, B)\Big) - \Big(S(A, B) + S(\overline{A}, \overline{B})\Big)\Big] + \frac{1}{2} = -\frac{1}{2}S_{new}(A, B) + \frac{1}{2} = 1 - Cohe(A, B)$.

(iii) $Cohe(\emptyset, X) = 0$.
$Cohe(\emptyset, X) = \frac{1}{4}\Big[\Big(S(\emptyset, X) + S(X, \emptyset)\Big) - \Big(S(X, X) + S(\emptyset, \emptyset)\Big)\Big] + \frac{1}{2}$. By condition $3'$ $S(\emptyset, X) = S(X, \emptyset) = 0$ and by condition $2'$ $S(X, X) = S(\emptyset, \emptyset) = 1$. It follows that $Cohe(\emptyset, X) = 0$. □

Note that Eq. 8 merely transforms S_{new} so that it lies in the interval $[0, 1]$ and so, in effect, Proposition 8 says that S_{new} is a coherence measure. The converse result does not hold, however, i.e. a coherence measure does not necessarily satisfy the propositions noted under Proposition 5 (even if it is transformed so that, like S_{new} it lies in the interval $[-1, 1]$). Nevertheless, it is clear that, like S_{new}, coherence measures have a neutral point such that for any fuzzy set A, $Cohe(A, I) = 0.5$.[2] Furthermore, it seems like a sensible extension of the idea behind coherence that coherence measures (in Sancho-Royo and Verdegay's sense) should satisfy Proposition 5(a) (i.e. should be type two similarity measures), even though this does not follow from their definition of coherence. This can be seen from the fact that measures defined in their paper, such as those based on distance metrics, are type two similarity measures.

The fact that the S_2 similarity measure as defined in Eq. 6 and similarity measures constructed using Eq. 7 share similar properties can now be expressed more formally by noting that S_2, in addition to being a type two similarity measure, is also a coherence measure as the following proposition asserts.

Proposition 9 *The S_2 similarity measure is a coherence measure.*

Proof

(i) $S_2(A, B) = S_2(B, A)$. Follows immediately from the definition.

(ii) $S_2(A, \overline{B}) = 1 - S_2(A, B)$.
We write $S_2(A, \overline{B}) = 1 - \sum_x v_x/|X|$ and $S_2(A, B) = 1 - \sum_x w_x/|X|$ where $v_x = max[min(\mu_A(x), \mu_B(x)), min(1-\mu_A(x), 1-\mu_B(x))]$ and $w_x = max[min(\mu_A(x), 1-\mu_B(x)), min(1-\mu_A(x), \mu_B(x))]$.
For a given $x \in X$ we can assume without loss of generality that

$$\mu_A(x) \leq \mu_B(x). \tag{9}$$

From this it clearly follows that $1 - \mu_B(x) \leq 1 - \mu_A(x)$. Clearly, either $\mu_A(x) \geq 1 - \mu_B(x)$ or $\mu_A(x) < 1 - \mu_B(x)$. We consider each case in turn.
Case 1 $\mu_A(x) \geq 1 - \mu_B(x)$ In this case $v_x = \mu_A(x)$. Now consider the corresponding term, w_x in the expression for $S_2(A, B)$. We know from the assumption in this case that $\mu_A(x) \geq 1 - \mu_B(x)$ and so $\mu_B(x) \geq 1 - \mu_A(x)$. From this it follows that $w_x = max(1 - \mu_B(x), 1 - \mu_A(x))$ and so it follows from Eq. 9 that $w_x = 1 - v_x$.

[2] This follows from the fact that $Cohe(A, I) = Cohe(A, \overline{I})$ and so from part (ii) of Definition 5 $Cohe(A, I) = 1 - Cohe(A, I)$.

Case 2 $\mu_A(x) < 1 - \mu_B(x)$ In this case $v_x = 1 - \mu_B(x)$. Now consider the corresponding term, w_x in the expression for $S_2(A, B)$. We know from the assumption in this case that $\mu_A(x) < 1 - \mu_B(x)$ and so $\mu_B(x) < 1 - \mu_A(x)$. From this it follows that $w_x = max(\mu_A(x), \mu_B(x))$ and so it follows from Eq. 9 that $w_x = 1 - v_x$. Thus, $\forall x \in X$, we know that $w_x = 1 - v_x$ and so

$$S_2(A, \overline{B}) = 1 - \frac{\sum_x v_x}{|X|} = 1 - \frac{\sum_x (1 - w_x)}{|X|} = \frac{\sum_x w_x}{|X|} = 1 - S_2(A, B). \quad (10)$$

(iii) $Cohe(\emptyset, X) = 0$. Follows immediately from the definition. □

This section has drawn attention to two very different types of similarity measures, which are analogous to the two types of coherence measures discussed in Sect. 2. One type is dominant in the literature and essentially ignores the degree of fuzziness of the sets. The other type incorporates fuzziness and corresponds very closely to the coherence measures proposed by Sancho-Royo and Verdegay. In particular, type two measures that have been constructed from type one measures via Eq. 7 can be converted into coherence measures via Eq. 8. Furthermore, the S_2 measure, which is a type two measure, is also a coherence measure.

4 Coherence of two knowledge bases

We now consider how the concept of coherence might be applied to knowledge bases. Rather than focussing on individual knowledge bases (possibly consisting of the union of two knowledge bases) to determine their properties and then using this information to order them in some way, the goal is to obtain a coherence measure to describe the relationship between two knowledge bases, including agreement between them. This coherence measure can then be used to order the knowledge bases. Throughout this section the knowledge bases are assumed to be individually consistent. In Sect. 4.1 we additionally assume that they are consistent with each other, while we do not make this assumption in Sect. 4.2.

In this section we use classical logic and consider a finite propositional language formed from a set of atoms, i.e. propositional symbols. We represent an interpretation of the language ω by the set of literals true in the interpretation, where a literal is either a propositional symbol p or its negation $\neg p$. Suppose K is a knowledge base consisting of a set of propositional formulae. Following Hunter (2004), we let $I(K)$ be the set of interpretations of K delineated by $Atoms(K)$, the atoms used in K, and let $M(K, L)$ be the set of models of K that are in $I(L)$. Thus, $M(K, L) = \{\omega \models \bigwedge K | \omega \in I(L)\}$, where $\models$ is classical and $\bigwedge K$ is the conjunction of the formulae in K.

4.1 Consistent knowledge bases

To start with we consider the case where knowledge bases K_1 and K_2 contain consistent information, i.e. $K_1 \cup K_2$ is consistent. Although this might seem trivial since in general we expect there to be some conflict between the two knowledge bases, this problem is important for three reasons. First, even in cases where information from different sources is consistent we still wish to rank pairs of sources in terms of the extent to which they agree with each other and it is not obvious how this should be done. Second, dealing with this case will prove instructive for tackling the more difficult problem where we wish to differentiate between cases of inconsistency. Third, as we shall see, the measures proposed here are precise parallels of those used in the probabilistic case discussed in Sect. 2 and so bring out the connection between the different areas where the concept of coherence can be applied.

Analogous to the necessary conditions for a probabilistic measure of coherence, the following conditions can be proposed as necessary conditions for a coherence measure for two individually consistent knowledge bases K_1 and K_2, denoted $C(K_1, K_2)$.

1. $C(K_1, K_2) = C(K_2, K_1)$,
2. $C(K_1, K_2)$ is maximal if K_1 and K_2 are logically equivalent, and
3. $C(K_1, K_2)$ is minimal if K_1 and K_2 are logically inconsistent.

Note that inconsistency is treated trivially according to these conditions since inconsistency always gives rise to a minimal value of coherence. Analogous to the probabilistic measure C_1, we define a coherence measure C_{KB1} for the coherence of two knowledge bases K_1 and K_2 in terms of the relative overlap between their models,

$$C_{KB1}(K_1, K_2) = \frac{|M(K_1, K_1 \cup K_2) \cap M(K_2, K_1 \cup K_2)|}{|M(K_1, K_1 \cup K_2) \cup M(K_2, K_1 \cup K_2)|}, \tag{11}$$

where we note that $M(K_1, K_1 \cup K_2)$ is the set of classical models of K_1 that are in the set of interpretations arising from the atoms contained in $K_1 \cup K_2$ and similarly for $M(K_2, K_1 \cup K_2)$.

Example 3 $C_{KB1}(\{p\}, \{p, q\}) = 1/2$, $C_{KB1}(\{p\}, \{p \vee q\}) = 2/3$, $C_{KB1}(\{p\}, \{q\}) = 1/3$ and $C_{KB1}(\{\neg p\}, \{p, q\}) = 0$.

In order to obtain a measure analogous to the probabilistic measure C_2 we first state Hunter's (2004) definition of degree of entailment (changing only the notation).

Definition 6 (*Hunter 2004*) Let X and Y be sets of classical propositional formulae each of which is consistent (i.e. $X \nvdash \perp$ and $Y \nvdash \perp$). The degree of entailment of X for Y, denoted $E(Y|X)$, is defined as follows,

$$E(Y|X) = \frac{|M(X, X \cup Y) \cap M(Y, X \cup Y)|}{|M(X, X \cup Y)|}. \tag{12}$$

The structure of Eq. 12 clearly indicates a correspondence between the degree of entailment of X for Y and conditional probability. We also need to define the negation of a knowledge base before obtaining a measure analogous to C_2.

Definition 7 Let K be a knowledge base and $M(K, K)$ the set of models of K delineated by the set of atoms in K. For each interpretation $i \in I(K)$, let $\alpha_i = \bigwedge_j l_{ij}$ where l_{ij} are the literals in interpretation i. We define the negation of K, denoted $\neg K$, to be

$$\neg K = \bigvee_{i \in I(K) \setminus M(K,K)} \alpha_i. \tag{13}$$

Note that the models of $\neg K$ are the set-theoretical complement of the models of K, i.e. $M(\neg K, K) = I(K) \setminus M(K, K)$.

Applying Definitions 6 and 7 to the probabilistic measure C_2 results in C_{KB2}, a measure of coherence for two knowledge bases K_1 and K_2 each of which is consistent and non-tautologous (i.e. $K_i \nvdash \perp$ and $K_i \nvdash \top$),

$$C_{KB2}(K_1, K_2) = \frac{1}{2}\left[\frac{E(K_2|K_1) - E(K_2|\neg K_1)}{E(K_2|K_1) + E(K_2|\neg K_1)} + \frac{E(K_1|K_2) - E(K_1|\neg K_2)}{E(K_1|K_2) + E(K_1|\neg K_2)}\right]. \tag{14}$$

Example 4 $C_{KB2}(\{p\}, \{p, q\}) = 3/4$, $C_{KB2}(\{p\}, \{p \vee q\}) = 2/3$, $C_{KB2}(\{p\}, \{q\}) = 0$ and $C_{KB2}(\{\neg p\}, \{p, q\}) = -1$.

Like the C_{KB1} measure the C_{KB2} measure satisfies conditions 1, 2 and 3, although its range is the interval $[-1, 1]$ rather than $[0, 1]$. Furthermore, various properties of these measures correspond with properties of the respective probabilistic measures of coherence. In particular, we note that just as zero provided a neutral point between positive and negative probabilistic dependence in the C_2 measure so zero provides a neutral point for the C_{KB2} measure. To make this point in a more precise way we define a notion of logical (in)dependence for knowledge bases.

Definition 8 For two knowledge bases K_1 and K_2, each of which is consistent and non-tautologous, we say that

(i) K_1 and K_2 are positively dependent iff $E(K_2|K_1) > E(K_2|\neg K_1)$,
(ii) K_1 and K_2 are independent iff $E(K_2|K_1) = E(K_2|\neg K_1)$,
(iii) K_1 and K_2 are negatively dependent iff $E(K_2|K_1) < E(K_2|\neg K_1)$.

The following proposition shows how logical (in)dependence affects coherence as given by the C_{KB2} measure.

Proposition 10 *For two knowledge bases K_1 and K_2, each of which is consistent and non-tautologous,*

(i) $C_{KB2}(K_1, K_2) > 0$ *iff K_1 and K_2 are positively dependent,*
(ii) $C_{KB2}(K_1, K_2) = 0$ *iff K_1 and K_2 are independent,*
(iii) $C_{KB2}(K_1, K_2) < 0$ *iff K_1 and K_2 are negatively dependent.*

Proof First of all, we show that

$$E(K_2|K_1) > E(K_2|\neg K_1) \text{ iff } E(K_1|K_2) > E(K_1|\neg K_2). \tag{15}$$

We just prove sufficiency since necessity can be proved in the same way. Let us define $E(K_2)$ as the ratio of the number of models of K_2 to the number of interpretations arising from the atoms in K_1 and K_2,

$$E(K_2) = \frac{|M(K_2, K_1 \cup K_2)|}{|I(K_1 \cup K_2)|}. \tag{16}$$

We note that $E(K_2)$ can be thought of as the marginal probability of K_2. It follows that

$$E(K_2) = E(K_2|K_1)E(K_1) + E(K_2|\neg K_1)E(\neg K_1). \tag{17}$$

Thus since $E(K_1) + E(\neg K_1) = 1$ and by assumption $E(K_2|K_1) > E(K_2|\neg K_1)$, it follows that

$$E(K_2|K_1) > E(K_2). \tag{18}$$

Furthermore, by the definitions given it follows that

$$E(K_1|K_2) = \frac{E(K_2|K_1)}{E(K_2)} E(K_1), \tag{19}$$

analogous to Bayes' theorem. Given Eq. 18 it follows that $E(K_1|K_2) > E(K_1)$. Considering the expression for $E(K_1)$ analogous to Eq. 17 it then follows that $E(K_1|K_2) > E(K_1|\neg K_2)$. From expression (15) Proposition (i) follows trivially. Propositions (ii) and (iii) can be established in a similar way. □

Thus, the neutral point in the C_{KB2} measure corresponds to logical independence of the knowledge bases and so we find that the coherence of two knowledge bases containing no atoms in common is zero,[3] e.g. $C_{KB2}(\{p\}, \{q, r\}) = 0$. Other properties of the C_{KB1} and C_{KB2} measures correspond to the properties of the two types of probabilistic coherence measures discussed in Sect. 2. However, we will not pursue these further since our main interest in this section is to consider cases where inconsistency is present.

4.2 Inconsistent knowledge bases

We now turn to the more demanding task of proposing coherence measures that will distinguish between cases where the two knowledge bases are inconsistent with each other, although we will still assume that they are individually consistent. The C_{KB1} and C_{KB2} measures defined in Sect. 4.2 will be inappropriate since they satisfy condition 3 and so yield minimal values of coherence (0 for C_{KB1} and -1 for C_{KB2}) in all cases where the knowledge bases are inconsistent. The goal in this section is to find a measure that will treat inconsistency in a non-trivial way, discriminating between different cases of inconsistency. We note, first of all, that such a measure can be obtained by a slight modification of the C_{KB2} measure. As can be seen from Eq. 14 the denominators ensure that the result is -1 in the case of inconsistency. By removing the denominators we obtain,

$$C_{KB3}(K_1, K_2) = \frac{1}{2}\left[E(K_2|K_1) - E(K_2|\neg K_1) + E(K_1|K_2) - E(K_1|\neg K_2)\right]. \tag{20}$$

We note that C_{KB3} satisfies conditions 1 and 2 proposed for a coherence measure in Sect. 4.1 and still results in values of coherence in the range [-1,1].

Example 5 $C_{KB3}(\{p\}, \{p, q\}) = 7/12$, $C_{KB3}(\{p\}, \{p \vee q\}) = 7/12$, $C_{KB3}(\{p\}, \{q\}) = 0$ and $C_{KB2}(\{\neg p\}, \{p, q\}) = -7/12$, $C_{KB3}(\{p, q\}, \{\neg p, \neg q\}) = -1/3$ and $C_{KB3}(\{p \to q, r\}, \{q \to \neg r, p \vee (\neg p \wedge \neg r)\}) = -1$.

As with the C_{KB2} measure, we find that C_{KB3} has a neutral point at zero corresponding to logical independence of the knowledge bases as defined in Definition 8. Furthermore, it is clear that C_{KB3} satisfies Proposition 10. While inconsistent knowledge bases will have a negative value of coherence according to C_{KB3} (since $E(K_2|K_1) = E(K_1|K_2) = 0$), it is worth noting that consistent knowledge bases can also have a negative coherence provided there is sufficiently little overlap between their sets of models. For example, $C_{KB3}(\{p\}, \{\neg p \vee \neg q\}) = -7/12$ even though they have a model in common, $\{p, \neg q\}$.

According to C_{KB3} the minimum value of coherence (-1) is obtained for knowledge bases containing complementary models, i.e. where one knowledge base is logically equivalent to the negation of the other as defined in Definition 7. Thus, condition 3 (see Sect. 4.1) has effectively been replaced by

3′. $C(K_1, K_2)$ is minimal if K_1 is the negation of K_2.

This, however, gives rise to a problem. Consider two conflicting predictions about tomorrow's weather: in scenario one, the first prediction is that it will be hot and dry and the second prediction is that it will be cold *or* wet; in scenario two, the first prediction is that it will be hot and dry and the second prediction is that it will be cold *and* wet. Intuitively it seems that scenario two is less coherent than scenario one since there is no possibility of even partial agreement between the predictions. This is not reflected in the C_{KB3} measure, however, since $C_{KB3}(\{p, q\}, \{\neg p, \neg q\}) = -1/3$ while $C_{KB3}(\{p, q\}, \{\neg p \vee \neg q\}) = -1$. The reason C_{KB3}

[3] The proof of this is exactly the same as that provided for the C_{KB3} measure in Sect. 4.2.

cannot account for this is that it does not take into account partial agreement between models of the two knowledge bases. Thus, there needs to be some contribution to the coherence measure from models that are not identical.

In light of the above discussion, we now define the notion of *categorical inconsistency* before providing an alternative to condition $3'$.

Definition 9 Consider two classical knowledge bases K_1 and K_2, each of which is individually consistent. K_1 and K_2 are defined to be categorically inconsistent if and only if K_1 and K_2 are categorical upon their common language and their models contain complementary literals, i.e. K_1 and K_2 each have just one model in their common language and for each atom p in the language, either p is in the model of K_1 and it is not in the model of K_2 or vice versa.

Example 6 The knowledge bases $K_1 = \{p, q, \neg r\}$ and $K_2 = \{\neg p, \neg q, r\}$ are categorically inconsistent.

The following alternative to condition $3'$ can now be defined,

$3''$. $C(K_1, K_2)$ is minimal if K_1 and K_2 are categorically inconsistent.

In order to construct a measure satisfying conditions 1, 2 and $3''$ and that includes a contribution from non-identical models, we first define the agreement between two classical interpretations, X and Y.

Definition 10 The agreement between two interpretations, X and Y is defined as

$$C(X, Y) = \frac{|X \cap Y|}{n}, \tag{21}$$

where n is the number of atoms in the language and we recall that interpretations are represented by the set of literals that are true in them.

Using Definition 10 and representing $M(K_i, K_1 \cup K_2)$ by $M(K_i)$, we can now define a coherence measure C_{KB4} between two knowledge bases K_1 and K_2 satisfying conditions 1, 2 and $3''$ as follows,

$$C_{KB4}(K_1, K_2) = \frac{|M(K_1) \cap M(K_2)| + \delta}{|M(K_1) \cup M(K_2)|}, \tag{22}$$

where

$$\delta = \frac{\sum_{X \in M_1} \sum_{Y \in M(K_2)} C(X, Y)}{|M(K_2)|} + \frac{\sum_{X \in M(K_1)} \sum_{Y \in M_2} C(X, Y)}{|M(K_1)|} \tag{23}$$

with $M_1 = M(K_1) \setminus M(K_2)$ and $M_2 = M(K_2) \setminus M(K_1)$. This means that for each model of K_1 which is not a model of K_2, we take its average agreement with the models of K_2 (and similarly for models of K_2 that are not models of K_1).

Example 7 $C_{KB4}(\{p\}, \{p, q\}) = 3/4$, $C_{KB4}(\{p\}, \{p \vee q\}) = 3/4$, $C_{KB4}(\{p\}, \{q\}) = 1/2$, $C_{KB4}(\{\neg p\}, \{p, q\}) = 1/4$, $C_{KB4}(\{p, q\}, \{\neg p, \neg q\}) = 0$ and $C_{KB4}(\{p \rightarrow q, r\}, \{q \rightarrow \neg r, p \vee (\neg p \wedge \neg r)\}) = 102/120$.

C_{KB4} defines a measure on the interval [0, 1], which satisfies conditions 1, 2 and $3''$. This measure is very closely associated with the C_{KB1} measure as can be seen by comparing Eq. 22 with Eq. 11. It is clear that in all cases $C_{KB4} \geq C_{KB1}$ since it adds in the extra component δ and we note that $\delta = 0$ if K_1 and K_2 are logically equivalent. However, while C_{KB1} is a simple measure corresponding directly to the probabilistic measure C_1, C_{KB4} is more complex. The motivation for the extra term δ arises from the idea that while C_{KB1} provides a good starting point for a coherence measure it fails to take into account partial agreement between models of the respective knowledge bases. This is crucial in cases of inconsistency since more partial agreement will result in a higher degree of coherence. Consider again predictions concerning tomorrow's weather and, in particular, the claim that it will be hot and dry. Clearly the prediction that it will be hot and wet is in greater agreement with this claim than is the prediction that it will be cold and wet, yet neither of these predictions has any models in common with the original prediction and so C_{KB1} yields a result of zero in both cases. By contrast, C_{KB4} finds the prediction that it will be hot and wet to be more coherent with the original claim (a value of 1/2 compared to zero for the prediction that it will be cold and wet).

It is also worth noting that the δ term results in more intuitive orderings of coherence values even when the two knowledge bases are consistent with each other. Consider again the claim that tomorrow's weather will be hot and dry (represented by $K_1 = \{p \wedge q\}$). Now consider the prediction that it will be hot ($K_2 = \{(p\}$) and an alternative prediction that it will be hot and dry *or* that it will be cold and wet ($K_3\{(p \wedge q) \vee (\neg p \wedge \neg q\}$). Note that K_2 is equivalent to the prediction that it will be hot and dry *or* that it will be hot and wet $((p \wedge q) \vee (p \wedge \neg q)$. Intuitively, it seems that the coherence between K_2 and K_1 should be greater than that between K_3 and K_1 since even the prediction of K_2 that conflicts with K_1 is in partial agreement with it. Yet C_{KB1} ignores this partial agreement and gives $C_{KB1}(K_2, K_1) = C_{KB1}(K_3, K_1) = 1/2$. By contrast, C_{KB4} captures this insight, giving $C_{KB4}(K_2, K_1) = 3/4 > 1/2 = C_{KB4}(K_3, K_1)$.

The fact that C_{KB4} satisfies condition $3''$ means that it overcomes the problem that was noted for the C_{KB3} measure. This can be seen by noting that $C_{KB4}(\{p, q\}, \{\neg p, \neg q\}) = 0$ while $C_{KB4}(\{p, q\}, \{\neg p \vee \neg q\}) = 1/3$.

The C_{KB4} and C_{KB3} measures have certain properties analogous to those of the probabilistic measures C_1 and C_2 respectively, as exemplified by the neutral point corresponding to probabilistic independence in the case of C_2 and to logical independence in the case of C_{KB3}. Furthermore, analogous to Proposition 3 (and to Proposition 7 dealing with similarity measures) we have the following proposition,

Proposition 11 *The C_{KB4} coherence measure defined in Eq. 22 is insensitive to the dependence relationship between knowledge bases as defined in Definition 8 in the sense that* $\forall \varepsilon \in (0, 1)$

(*a*) *there are knowledge bases K_1 and K_2 which have a negative dependence and C_{KB4} $(K_1, K_2) > 1 - \varepsilon$,*

(*b*) *there are knowledge bases K_1 and K_2 which have a positive dependence and C_{KB4} $(K_1, K_2) < \varepsilon$.*

Proof

(a) Select an integer m such that $n = 2^m > 2/\varepsilon$. Now consider a language with propositional symbols $p_1, \ldots, p_m$ and let $K_1 = \{p_1 \vee p_2 \vee \cdots \vee p_m\}$ and $K_2 = \{\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_m\}$. We note that $E(K_2|K_1) = (n-2)/(n-1) < 1 = E(K_2|\neg K_1)$ and so K_1 and K_2 are negatively dependent according to Definition 8. We find that $C_{KB4}(K_1, K_2) > (n-2)/n$. (In fact, $(n-2)/n$ is the value of $C_{KB4}(K_1, K_2)$ if the δ term is excluded.) It follows that $C_{KB4}(K_1, K_2) > 1 - \varepsilon$.

(b) Select an integer n such that $n > 2/\varepsilon$. Now consider a language with propositional symbols $p_1, \ldots, p_n$ and let $K_1 = \{p_1 \wedge p_2 \wedge \cdots \wedge p_n\}$ and $K_2 = \{(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \vee \alpha\}$ where α is the propositional formula $\bigvee_{i=1,n-1}(\neg p_1 \wedge \cdots \wedge \neg p_{i-1} \wedge p_i \wedge$

$\neg p_{i+1} \wedge \cdots \wedge \neg p_n$). We note that $E(K_2|K_1) = 1 > (n-1)/(2^n - 1) = E(K_2|\neg K_1)$ and so K_1 and K_2 are positively dependent according to Definition 8. We find that $C_{KB4}(K_1, K_2) = 2/n - 1/n^2 < 2/n < \varepsilon$. □

The following example illustrates part (a) of Proposition 11.

Example 8 Suppose p is the proposition that 'John will pass the test' and q and r are the corresponding propositions for Mary and Tom respectively. Consider the claim K_1 that at least one of them will pass the test ($p \vee q \vee r$) and the claim K_2 that at least one of them will fail the test ($\neg p \vee \neg q \vee \neg r$). In this case the coherence according to C_{KB4} is high (6/7) even though $E(K_2|K_1) = 6/7 < 1 = E(K_2|\neg K_1)$ and so K_1 and K_2 are negatively dependent. The fact that most outcomes are compatible with K_1 and K_2 results in a high degree of coherence since the first term in C_{KB4} is just the relative overlap of the models of K_1 and K_2.

It is obvious from the definition of the C_{KB3} measure in Eq. 20 that it does not satisfy a corresponding version (taking into account its different range) of Proposition 11 since it is constructed to ensure that negative (positive) dependence results in a value of C_{KB3} that is negative (positive).

Intuitively it might be thought that two knowledge bases having no atoms in common should be neither coherent nor incoherent, i.e. should have a neutral value of 0 for the C_{KB3} measure and $1/2$ for the C_{KB4} measure, since they are providing information about different things. This is the case for the C_{KB3} measure as the following proposition shows.

Proposition 12 *Suppose that two knowledge bases K_1 and K_2 have no atoms in common so that $Atoms(K_1) \cap Atoms(K_2) = \varnothing$. Then, $C_{KB3}(K_1, K_2) = 0$.*

Proof From Proposition 10 it is sufficient to show that K_1 and K_2 are independent as defined in Definition 8. Given Eq. 17 this is equivalent to showing that $E(K_2|K_1) = E(K_2)$.

The number of models of K_1 in the language consisting of the atoms in $K_1 \cup K_2$ will be the number of models of K_1 in its own language times the number of interpretations in the language of K_2 (i.e. the language consisting of the atoms in K_2). This can be written as $|M(K_1, K_1 \cup K_2)| = |I(K_2)| \times |M(K_1, K_1)|$. The number of models of K_1 in the joint language that are also models of K_2 is $|M(K_1, K_1)| \times |M(K_2, K_2)|$. Thus,

$$E(K_2|K_1) = \frac{|M(K_1, K_1)| \times |M(K_2, K_2)|}{|I(K_2)| \times |M(K_1, K_1)|} = \frac{|M(K_2, K_2)|}{|I(K_2)|} = E(K_2). \tag{24}$$

This completes the proof. □

Only a more limited version of Proposition 12 holds for the C_{KB4} measure.

Proposition 13 *Suppose that K_1 is the knowledge base $\{p_1, \ldots, p_m\}$ and $K_2 = \{p_{m+1}, \ldots, p_{m+n}\}$. Then, $C_{KB4}(K_1, K_2) = 1/2$.*

Proof First of all, we note that the relevant models of K_1 and K_2 will be a subset of the interpretations in $I(K_1 \cup K_2)$, the set of interpretations arising from the atoms in K_1 and K_2. Clearly, there will be only one model of K_1 that is also a model of K_2, i.e. $\{p_1, \ldots, p_{m+n}\}$. Furthermore, there will be 2^n models of K_1, i.e. one for each model of K_2 in $I(K_2)$, and 2^m models of K_2. Thus, from Eq. 22 the coherence can be written as $C_{KB4}(K_1, K_2) = (1+\delta)/(2^m + 2^n - 1)$. Hence, it suffices to show that $\delta = (2^m + 2^n - 3)/2$.

Consider first of all the contribution to δ from models of K_1 that are not models of K_2. For each such model, its contribution will be an average over all 2^m models of K_2 as expressed in Eq. 23. Let us consider a model of K_1 that differs in j literals corresponding to the atoms in K_2 from the models of K_2. By considering the sum as contributions from models of K_2 differing from the model in question by k literals corresponding to the atoms in K_1 we can write the summation in terms of a sum of binomial coefficients times the ratio of the atoms in agreement to the total number of atoms, i.e.,

$$\frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}\frac{m-k+n-j}{m+n}. \tag{25}$$

We then need to include a further summation over j for all the models of K_1 which need to be included and again we find that a binomial coefficient is required. After adding a similar term for models of K_2 that are not models of K_1, we obtain the following expression for δ

$$\delta = \frac{1}{2^m}\sum_{j=1}^{n}\sum_{k=0}^{m}\binom{n}{j}\binom{m}{k}\frac{m-k+n-j}{m+n} + \frac{1}{2^n}\sum_{j=1}^{m}\sum_{k=0}^{n}\binom{m}{j}\binom{n}{k}\frac{n-k+m-j}{m+n}. \tag{26}$$

By noting that $\sum_{k=0}^{m}\binom{m}{k}(m-k) = m2^{m-1}$ we are able to rewrite the expression for δ as

$$\begin{aligned}\delta &= \frac{1}{2^m(m+n)}\left[2^{m-1}m(2^n-1)+2^m n(2^{n-1}-1)\right]\\ &\quad+\frac{1}{2^n(m+n)}\left[2^{n-1}n(2^m-1)+2^n m(2^{m-1}-1)\right]\\ &= \frac{2^n+2^m-3}{2}.\end{aligned} \tag{27}$$

This completes the proof. □

It is worth noting that the C_{KB4} measure does not satisfy a stronger claim corresponding to Proposition 12, i.e. that its value equal 1/2 in cases where the two knowledge bases have no atoms in common, as the following example shows.

Example 9 Let $K_1 = \{p \leftrightarrow q\}$ and $K_2 = \{r\}$. $C_{KB4}(K_1, K_2) = 11/18$.

The way in which C_{KB4} differs from C_{KB3} in this respect is similar to the way in which the probabilistic coherence measures C_1 and C_2 differ since the proof of Proposition 12 shows that knowledge bases with no atoms in common are logically independent. In the probabilistic case, C_2 gives a value of zero in the case of probabilistic independence, just as C_{KB3} does in the case of logical independence. By contrast, consider the C_1 measure for a case of probabilistic independence in which the two beliefs have high marginal probabilities. In this case, C_1 can have a high value (close to 1) since it is the overlap that is important, not probabilistic (in)dependence. Similarly, for the C_{KB4} measure it is overlap/agreement between the models of the two knowledge bases that is important, not logical (in)dependence. Hence, if both knowledge bases have a large number of models (which corresponds to high marginal probability in the probabilistic case) the C_{KB4} measure can have a high value (close to 1).

In comparing the two coherence measures presented in this section (C_{KB3} and C_{KB4}), we note that each of them has some advantages. Despite the fact that the C_{KB3} measure is straightforward, being based on the degree of entailment, and presents a clear parallel to the probabilistic measure C_2, the fact that it does not have a minimal value of coherence for the knowledge bases $K_1 = \{p, q\}$ and $K_2 = \{\neg p, \neg q\}$ presents a serious problem. Hence, overall C_{KB4} seems preferable as a measure of coherence and so only it will be considered in the following discussion.

Finally in this section, we consider the issue of how pairs of knowledge bases can be ordered in terms of the coherence measures that have been defined. To do this we propose a definition of closeness of two knowledge bases.

Definition 11 Let K, K_1 and K_2 be three knowledge bases. K_1 is defined as being closer than K_2 to K, denoted $K_2 \preceq_K K_1$, iff

$$C_{KB4}(K_2, K) \leq C_{KB4}(K_1, K). \tag{28}$$

Example 10 Let $K = \{\neg p\}$, $K_1 = \{p, q\}$ and $K_2 = \{p \vee q\}$.K_2 is closer to K since $C_{KB4}(K_1, K) = 1/4 < 11/24 = C_{KB4}(K_2, K)$.

4.3 Discussion of related work

The coherence measures proposed above to deal with two knowledge bases which may be inconsistent with each other draw attention to the fact that it is not just the degree of conflict between them that is important when comparing them: the extent of agreement can be an important factor as well. This is illustrated by a problem Qi et al. (2005) raise concerning a measure of the degree of conflict between two knowledge bases proposed by Hunter (2004). They point out that the measure yields the degree of conflict between $\{p, q, r\}$ and $\{\neg p, q, r\}$ to be the same as that between $\{p, q, r\}$ and $\{\neg p\}$ (a value of 1/3), whereas one would expect the agreement on literals q and r in the first case would result in a lower degree of conflict.[4] One response to this criticism would be to say that although there is clearly a difference in the amount of agreement in the two cases, there is no difference in the degree of conflict. Nevertheless, whether this response is accepted or not, it does seem appropriate to provide a measure that takes the agreement into account and so incorporates the point raised by Qi et al. This is achieved by the C_{KB4} measure, which deals with this example in an appropriate way since $C_{KB4}(\{p, q, r\}, \{\neg p, q, r\}) = 2/3 > 1/3 = C_{KB4}(\{p, q, r\}, \{\neg p\})$ and so the first pair of knowledge bases are more coherent than the second pair according to C_{KB4}.

As already noted, the work of Qi et al is closely related to the current work. Although their work deals with prioritised knowledge bases, we only consider here the limiting case where the weights are one and so the knowledge bases are classical. As in the current work, they consider the case of two individually consistent knowledge bases and propose a degree of conflict between them which takes into account agreement between the knowledge bases and so overcomes the problems noted in the example above. Consider two knowledge bases K_1 and K_2 and let C and D be prime implicants[5] of K_1 and K_2 respectively. They define a degree of conflict between C and D, which in the classical case can be written as,

$$d_{Con}(C, D) = \frac{Atom_C(C, D)}{Atom_C(C, D) + Atom_{SA}(C, D) + \lambda \times Atom_{WA}(C, D)}, \tag{29}$$

[4] Hunter (2002) defines the coherence of a quasi-classical model and extends this to the coherence of a knowledge base by taking the maximum coherence of all its minimal quasi-classical models. However, this definition of coherence is intended as a measure of inconsistency and, being applied to a single knowledge base, is not intended to take into account the agreement between two knowledge bases. To apply this approach to the current problem, it would be necessary to take the union of the two knowledge bases which in both cases would result in $\{p, q, r, \neg p\}$. Hence, in both cases the coherence would be the same.

[5] Following the definition by Qi et al. (2005) we say that a conjunction of literals D is an implicant of formula ϕ iff $D \models \phi$ and D does not contain two complementary literals. A prime implicant of a knowledge base K is an implicant D of K such that for every other implicant D' of K, $D' \not\subset D$.

where $Atom_C(C, D)$ is the cardinality of the set of atoms in conflict[6] in $C \cup D$, $Atom_{SA}(C, D)$ is the cardinality of the set of atoms which are included in both C and D (they are said to be in strong agreement), $Atom_{WA}(C, D)$ is the cardinality of the set of atoms which are included in C or D but not both (they are said to be in weak agreement) and λ can take any value in the interval (0, 1]. Here we will take $\lambda = 1/2$. The degree of conflict between K_1 and K_2, denoted $D_{Con}(K_1, K_2)$ is then taken to be the minimum degree of conflict between pairs of prime implicants taken from K_1 and K_2 respectively. For the example discussed earlier, this yields $D_{Con}(\{p, q, r\}, \{\neg p, q, r\}) = 1/3 < 1/2 = D_{Con}(\{p, q, r\}, \{\neg p\})$, and so overcomes the problem.

A problem arises for the approach of Qi et al. when the degree of conflict is used to define the closeness of two knowledge bases. Briefly, knowledge bases K_1 and K are closer than K_2 and K if and only if the degree of conflict is greater between K_2 and K than it is between K_1 and K. Consider the knowledge bases $K = \{\neg p, q\}$, $K_1 = \{p, q\}$, $K_2 = \{p\}$ and $K_3 = \{p, r\}$. Since $D_{Con}(K_1, K) = 1/2 < 2/3 = D_{Con}(K_2, K)$ and so K_1 and K are closer than K_2 and K. However, $D_{Con}(K_1, K) = 1/2 = D_{Con}(K_3, K)$ and so K_3 is as close to K as K_1 is to K. Yet, this seems unintuitive since there is clearly more agreement between K_1 and K than there is between K_3 and K. In considering K_3 and K, the atoms q and r are said to be in weak agreement and, for the purposes of calculating the degree of conflict, this is equivalent to having one atom in strong agreement as is the case between K_1 and K. It is not clear, however, why weak agreement should be able to compensate for lack of strong agreement in this way. In fact, if there is sufficient weak agreement it can more than compensate for strong agreement. Considering $K_4 = \{p, r, s\}$, we find that $D_{Con}(K_4, K) = 2/5$ and so K_4 is closer to K than K_1 is to K, which again seems incorrect since intuitively K_1 has more agreement with K.

The same problems occur even if the value of λ is changed. If $\lambda > 1/2$ the problem becomes worse since weak agreement is able to compensate for strong agreement more easily. If $\lambda < 1/2$ the problem is lessened to some extent since weak agreement cannot compensate for strong agreement as easily as it can in the case where $\lambda = 1/2$. However, sufficient weak agreement will still be able to compensate for strong agreement, it is just that what counts as sufficient will change.

A more satisfactory ordering can be achieved by using the C_{KB4} coherence measure and the definition of closeness provided in Definition 11. Considering again knowledge bases K, K_1, K_2, K_3 and K_4, we find that $C_{KB4}(K_1, K) = 1/2, C_{KB4}(K_2, K) = 1/4, C_{KB4}(K_3, K) = 1/3$ and $C_{KB4}(K_4, K) = 3/8$. Thus, in terms of their closeness to K we find that K_1 is closer than K_4, which in turn is closer than K_3, which in turn is closer than K_2. Intuitively, this seems more reasonable. Adding more atoms to one knowledge base which are not in the other might result in a slight increase in coherence since the conflicting information is relatively less important to all the information in the knowledge bases, but this cannot compensate for lack of strongly agreeing information (i.e. atoms that are in both knowledge bases). This can be expressed more formally by noting that $C_{KB4}(\{\neg p, q\}, \{p, p_1, \ldots, p_m\} = 1/2 \times (m+1)/(m+2)$ which is less than $1/2 (= C_{KB4}(\{\neg p, q\}, \{p, q\}))$ for all finite m.

Since Qi et al. also define degrees of strong agreement and weak agreement, they could point out in response to the foregoing discussion that K_1 and K have a greater (lesser) degree of strong (weak) agreement than K_3 and K. While this is correct, the problem is that only the degree of conflict is taken into account in their definition of closeness. Suppose, however, that instead of defining closeness in terms of their degree of conflict it is defined in terms

[6] An atom p is in the set of atoms in conflict in $C \cup D$ iff p is in C and $\neg p$ is in D or vice versa.

of their degree of strong agreement. In the classical case the degree of strong agreement between two prime implicants C and D of knowledge bases K_1 and K_2 respectively can be written as

$$d_{SA}(C, D) = \frac{Atom_{SA}(C, D)}{Atom_C(C, D) + Atom_{SA}(C, D) + \lambda \times Atom_{WA}(C, D)}, \tag{30}$$

with the degree of strong agreement between K_1 and K_2, denoted $D_{SA}(K_1, K_2)$, taken to be the maximum degree of strong agreement between pairs of prime implicants taken from K_1 and K_2 respectively. Closeness of two knowledge bases could then be ordered by saying that K_1 and K are closer than K_2 and K if and only if the degree of strong agreement is greater between K_1 and K than it is between K_2 and K. This would resolve the problem for the knowledge bases considered above, since $D_{SA}(K_1, K) = 1/2$ while $D_{SA}(K_2, K) = D_{SA}(K_3, K) = D_{SA}(K_4, K) = 0$. However, it would also give rise to other problems. Just as weak agreement was found to compensate for lack of strong agreement in the ordering based on the degree of conflict, so weak agreement can now compensate for lack of conflicting information to *reduce* the degree of strong agreement (and hence closeness). For example, consider again $K = \{\neg p, q\}$, $K_1 = \{p, q\}$ and now also consider $K_5 = \{q, r, s\}$. Here we find that $D_{SA}(K_1, K) = 1/2 > 2/5 = D_{SA}(K_5, K)$ and so K_1 is closer than K_5 to K, which seems unintuitive since there is more disagreement between K_1 and K than there is between K_5 and K.

Once again, a more satisfactory ordering can be achieved by using the C_{KB4} coherence measure since $C_{KB4}(K_1, K) = 1/2 < 5/8 = C_{KB4}(K_5, K)$ and so K_5 is closer than K_1 to K. Complementary to the earlier case, adding more atoms to one knowledge base which are not in the other might result in a slight decrease in coherence since the agreeing information is relatively less important to all the information in the knowledge bases, but this cannot compensate for lack of conflicting information in reducing coherence. This can be expressed more formally by noting that $C_{KB4}(\{\neg p, q\}, \{q, p_1, \ldots, p_m\} = 1/2 \times (m + 3)/(m + 2)$ which is greater than $1/2 (= C_{KB4}(\{\neg p, q\}, \{p, q\}))$ for all finite m.

Despite the problems noted above, the work of Qi et al. shows that taking agreement into account, rather than just conflicting information, can be important. In fact, since orderings based on the degree of strong agreement overcome some problems encountered by the approach based on the degree of conflict, this suggests that they would need to incorporate *both* the degrees of strong agreement and conflict (and perhaps also their degree of weak agreement) in their definition of closeness. By contrast, the C_{KB4} coherence measure provides a single, integrated way of incorporating both agreeing and conflicting information that overcomes the problems discussed.

5 Conclusions

In this paper we have discussed differences between two probabilistic measures of coherence and explored parallels between this work, similarity measures for fuzzy sets and the inconsistency between two knowledge bases. In the case of similarity, two very different conceptions of similarity have been identified, one ignoring the degree of fuzziness of the sets and the other incorporating it. It has been shown how measures of the first type can be converted to measures of the second type and that the second type is closely related to the notion of a coherence measure proposed by Sancho-Royo and Verdegay (2000). Furthermore, a notion of fuzzy-independence was defined relative to a similarity measure of the first type and it was shown that when such a similarity measure is used to generate a similarity measure of the second type, the former measure is insensitive to fuzzy-dependence while the latter is not. This parallels the situation with the probabilistic measures of coherence.

It has also been argued that in comparing knowledge bases, it is often important to take agreement between them into account in addition to the total information and conflicting information. The idea has been to construct a coherence measure which incorporates each of these components. In the case where two knowledge bases are consistent with each other the probabilistic coherence measures can be carried across directly to the models of the knowledge bases, but in the case where the knowledge bases are in conflict this is not possible. Two measures were proposed for the inconsistent case and, once again, parallels with the probabilistic measures were found. In particular, as in the probabilistic and fuzzy cases, a notion of logical independence of two knowledge bases was defined, with one measure insensitive to independence while the other is not. However, the measure that resulted in a positive (negative) value in cases of positive (negative) logical dependence turned out to be problematic since it does not yield as minimally coherent (or maximally incoherent) knowledge bases which intuitively it should. By contrast, the coherence measure that is insensitive to logical (in)dependence satisfies appropriate conditions for a measure of coherence, has a number of suitable properties and has been shown to yield intuitive results in comparison of knowledge bases which present problems for other approaches.

The application of coherence to inconsistency in knowledge bases suggests a number of possible directions for future work. These include treating cases where the knowledge bases may be individually inconsistent and extending the work to calculate coherence in prioritised knowledge bases as Qi et al. have done with their work on strong and weak agreement. Another direction would be to investigate how coherence measures can be applied to deal with interactions between agents and the ordering of heterogeneous sources of information.

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